Solving Differential Equations in Finance and Economics
Solution To Asset Price Stochastic Differential Equation

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A stochastic differential equation (SDE) is comprised of both a deterministic (time) and a random (Brownian motion) component. The following equation is a standard differential equation that describes the change in asset price...

$$\delta S_t = \mu\, S_t\, \delta t + \sigma\, S_t\, \delta W_t \tag{1}$$

In Equation (1) above the variable S_t is asset price at time t, the variable μ is the expected rate of return, the variable σ is return volatility, the variable δt is the change in time (deterministic), and the variable δW_t is the change in the underlying Brownian motion (random). This equation should be interpreted as the change in asset price is comprised of an expected return, which is known at time t, plus an innovation, which is purely random and therefore is unknown at time t.

Our task in this white paper is to solve Equation (1) above.

Solving the SDE

Our first task is to determine the basic form of the equation that solves Equation (1) above. Note the following function and derivative...

$$F(t) = C\,\mathrm{Exp}\left\{\alpha\, t\right\} \text{ ...where... } \frac{\delta F(t)}{\delta t} = \alpha\, C\,\mathrm{Exp}\left\{\alpha\, t\right\} = \alpha\, F(t) \text{ ...such that... } \delta F(t) = \alpha\, F(t)\, \delta t \tag{2}$$

Note that the derivative of Equation (2) above has the basic form of Equation (1) above, which is the stochastic differential equation that we need to solve. Now that we have the basic form of the solution equation let's change Equation (2) above to be a function of asset price rather than time. The new equation is...

$$F(S_t) = \ln\left(S_t\right) \text{ ...such that... } \frac{\delta F(S_t)}{\delta S_t} = \frac{1}{S_t} \text{ ...and... } \frac{\delta F^2(S_t)}{\delta S_t^2} = -\frac{1}{S_t^2} \tag{3}$$

Using Equation (3) above the second order Taylor Series Expansion is...

$$\delta F(S_t) = \frac{1}{S_t}\,\delta S_t - \frac{1}{2}\,\frac{1}{S_t^2}\,\delta S_t^2 \tag{4}$$

Note that for Equation (4) above we have the equation for δS_t but not the square of δS_t. Using Equation (1) above the equation for the square of the change in asset price is...

$$(\delta S_t)^2 = (\mu\, S_t)^2\,(\delta t)^2 + (\sigma\, S_t)^2\,(\delta W_t)^2 + 2\,\mu\,\sigma\, S_t^2\,\delta t\,\delta W_t \tag{5}$$

Note the following product equations...

$$\text{As the change in time goes to zero... } (\delta t)^2 = 0 \text{ ...and... } (\delta W_t)^2 = \delta t \text{ ...and... } \delta t\,\delta W_t = 0 \tag{6}$$

Using Equation (6) above we can rewrite Equation (5) above as...

$$(\delta t)^2 = (\mu\, S_t)^2 \times 0 + (\sigma\, S_t)^2 \times t + 2\,\mu\,\sigma\, S_t^2 \times 0 = \sigma^2 S_t^2\,\delta t \tag{7}$$

Using Equations (1) and (7) above we can rewrite Equation (4) above as...

$$\delta F(S_t) = \frac{1}{S_t}\left(\mu\, S_t\,\delta t + \sigma\, S_t\,\delta W_t\right) - \frac{1}{2}\,\frac{1}{S_t^2}\left(\sigma^2 S_t^2\,\delta t\right) = \left(\mu - \frac{1}{2}\,\sigma^2\right)\delta t + \sigma\, S_t\,\delta W_t \tag{8}$$

Using Equation (8) above the equation for the log of asset price at time t as a function of the log of asset price at time zero (noting that the value of the Brownian motion at time zero is zero) can be written as...

$$F(S_t) = F(S_0) + \int\limits_0^t \delta S_u = F(S_0) + \int\limits_0^t \left(\mu - \frac{1}{2}\,\sigma^2\right)\delta u + \int\limits_0^t \sigma\,S_t\,\delta W_t \text{ ...where... } W_0 = 0 \tag{9}$$

The solutions to the two intergrals in Equation (9) above are...

$$\int\limits_0^t \left(\mu - \frac{1}{2}\,\sigma^2\right)\delta u = \left(\mu - \frac{1}{2}\,\sigma^2\right)t \text{ ...and... } \int\limits_0^t \sigma\,S_t\,\delta W_t = \sigma\,W_t \tag{10}$$

Using Equation (10) above we can rewrite Equation (9) above as...

$$F(S_t) = F(S_0) + \left(\mu - \frac{1}{2}\,\sigma^2\right)t + \sigma\,W_t \tag{11}$$

Equation (11) above is the equation for the log of asset price at time t. If we take the exponential of both sides of Equation (11) then the equation for asset price at time t, which is the solution to our stochastic differential equation, becomes...

$$\begin{aligned}
\text{Exp}\left\{F(S_t)\right\} &= \text{Exp}\left\{F(S_0) + \left(\mu - \frac{1}{2}\,\sigma^2\right)t + \sigma\,W_t\right\} \\
\text{Exp}\left\{F(S_t)\right\} &= \text{Exp}\left\{F(S_0)\right\}\text{Exp}\left\{\left(\mu - \frac{1}{2}\,\sigma^2\right)t + \sigma\,W_t\right\} \\
S_t &= S_0\,\text{Exp}\left\{\left(\mu - \frac{1}{2}\,\sigma^2\right)t + \sigma\,W_t\right\} \qquad (12)
\end{aligned}$$

So to summarize the solution to Equation (1) above is...

$$\text{The solution to the equation } \delta S_t = \mu\,S_t\,\delta t + \sigma\,S_t\,\delta W_t \text{ is } S_t = S_0\,\text{Exp}\left\{\left(\mu - \frac{1}{2}\,\sigma^2\right)t + \sigma\,W_t\right\} \tag{13}$$